

New stationary vacuum black string solution

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(Dated: February 15, 2007)

We find a new 4 + 1 dimensional stationary spherically symmetric vacuum black string solution which is invariant under the translation along the fifth coordinate. The new solution is characterized by three parameters, mass, tension, and conserved momentum along the fifth coordinate. The metric is locally equivalent to and globally different from the known static solution. We briefly discuss its physical properties.

PACS numbers: 04.70.-s, 04.50.+h, 11.25.Wx, 11.27.+d

I. INTRODUCTION

Recently, a generalized static black string solution with arbitrary tension τ and mass density M in (4+1)-dimensions has been found by Lee [1]. In fact, the solution was mentioned by other authors in eighties in a different context of Kluza-Klein gauge interaction [2, 3]. These higher dimensional solutions have become objects of serious consideration in physics such as the string theory [4] and brane cosmology [5, 6].

In studying these higher dimensional black string solutions, their stability becomes an interesting topic [7, 8]. Various black strings were shown to be unstable under small perturbations [7, 9, 10]. The possibility of the unstable black string finally fragmenting into black holes was mentioned [11]. However, Horowitz and Maeda [12] argued that event horizons could not pinch off.

In Ref. [1], the asymptotic properties of the metric up to $O(1/r)$ in asymptotic region was investigated and the static metric was studied. In this stage, it is interesting to search for a stationary solution with conserved momentum in the fifth coordinate. In fact, if there is an object with momentum, the velocity frame dragging effects do seem to appear in general relativity. The velocity frame dragging is not well known and controversial. In addition, its effects may alter local or global geometry. In this sense, it needs to study the effect of a conserved momentum on the geometry of stationary spacetime.

In this work, the static black string solution [1] is briefly reviewed in Sec. II. A general class of black string solutions with arbitrary tension and conserved momentum is presented in Sec. III. The local equivalence and global difference to the static black string solution are analyzed in Sec. IV. Physical properties of the solution is briefly discussed in Sec. V. In Sec. VI, we discuss the geometric properties of the horizon area, and the global difference of the metric.

II. BRIEF REVIEW ON THE STATIC BLACK STRING SOLUTION

After the reappearance of the static black string solution [1], its geometric properties are being studied [13] especially in relation to the tension of the string. The metric of the static spherically symmetric vacuum black string solution in (4+1) dimensions in Ref. [1] is

$$ds^2 = - \left| \frac{1 - K/\rho}{1 + K/\rho} \right|^{\frac{2(\chi+1/\sqrt{3})}{\sqrt{1+\chi^2}}} dt^2 + \left| \frac{1 - K/\rho}{1 + K/\rho} \right|^{\frac{2(-\chi+1/\sqrt{3})}{\sqrt{1+\chi^2}}} dz^2 \\ + \left| 1 - \frac{K^2}{\rho^2} \right|^2 \left| \frac{1 - K/\rho}{1 + K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2), \quad (1)$$

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where

$$K = \frac{G_5}{2\sqrt{3}}(M + \tau)\sqrt{1 + \chi^2}, \quad \chi = \sqrt{3}\frac{M - \tau}{M + \tau} = \sqrt{3}\frac{1 - a}{1 + a}, \quad (2)$$

with $a = \tau/M$. Here G_5 is the five dimensional gravitational constant. Let us briefly summarize the properties of the metric.

- The horizon (in the sense that no energy can crawl out of the $\rho = K$ surface from the singularity) is present at $\rho = K$ for $\chi > -\frac{1}{\sqrt{3}}$ or $\tau < 2M$.
- $\rho = K$ is a curvature singularity for $|\chi| \neq \frac{1}{\sqrt{3}}$.
- The solution describes a naked singularity at $\rho = K$ for $\chi < -\frac{1}{\sqrt{3}}$; ($a > 2$).
- The singularity is hidden by the horizon for $\chi > -\frac{1}{\sqrt{3}}$, therefore, the solution denotes a black string solution.
- The strong energy condition restricts $a \leq 2$ ($\chi \geq -1/\sqrt{3}$). Therefore, the curvature singularity is hidden behind the horizon for physical range of parameters $0 \leq a \leq 2$ ($-1/\sqrt{3} < \chi \leq \sqrt{3}$).
- Solution with $\chi = 1/\sqrt{3}$ and $\chi = -1/\sqrt{3}$ denote the Schwarzschild black string and the Kluza-Klein bubble solution [14].

Not all physical properties of the solution is yet understood and is under investigation [13]. This solution was found by noting that two independent asymptotic quantities are needed to characterize the static vacuum black string solution in 5-dimension [15], [16].

Let us generalize the discussion on the asymptotic quantities used in Ref. [1] to a stationary case in $(4 + 1)$ dimensional spacetime with the coordinates $x^0 = t$, x^i ($i = 1, 2, 3$), and $x^4 = z$. For the weak gravitational field produced by stationary z -independent source, the linearized Einstein equation in harmonic coordinates is

$$\partial_i \partial^i h_{\mu\nu} = -16\pi G_5 \bar{T}_{\mu\nu}, \quad (3)$$

where

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \quad (|h_{\mu\nu}| \ll 1),$$

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T.$$

Using the Green's function for the three dimensional Laplacian, the solution of Eq. (3) can be written by

$$h_{\mu\nu}(x) = 4G_5 \int d^3y \frac{\bar{T}_{\mu\nu}(y)}{|\mathbf{x} - \mathbf{y}|}.$$

The leading term of $h_{\mu\nu}(x)$ are then calculated to be

$$h_{00} \simeq \frac{4G_5}{r} \int d^3y \left(\frac{2}{3}T_{00} + \frac{1}{3}T_{44} \right),$$

$$h_{04} \simeq \frac{4G_5}{r} \int d^3y T_{04},$$

$$h_{ij} \simeq \frac{4G_5 \delta_{ij}}{3r} \int d^3y (T_{00} - T_{44}),$$

$$h_{44} \simeq \frac{4G_5}{r} \int d^3y \left(\frac{1}{3}T_{00} + \frac{2}{3}T_{44} \right),$$

where $r = \sqrt{\sum_{i=1}^3 x_i^2}$.

For the stationary case with $T^{04} \neq 0$, the leading corrections to the metric far away from the source, up to the order of $1/r$, can be seen to be characterized by the three quantities

$$M \equiv \int d^3x T_{00}, \quad \tau \equiv \int d^3x T_{44}, \quad P \equiv \int d^3x T_{04}, \quad (4)$$

where M is the mass per unit length, τ is the tension, and P is the conserved momentum along the z -direction of the source. For later convenience, we use the parameter j ,

$$j = \frac{2P}{M - \tau}, \quad (5)$$

in prefer of P . Then, up to the order of $1/r$,

$$h_{00}(x) \simeq \frac{4G_5}{3} \frac{2M - \tau}{r}, \quad h_{ij}(x) \simeq \frac{4G_5}{3} \delta_{ij} \frac{M + \tau}{r}, \quad h_{44}(x) \simeq \frac{4G_5}{3} \frac{M - 2\tau}{r}, \quad h_{04}(x) \simeq \frac{2G_5(M - \tau)j}{r}. \quad (6)$$

If the coordinate z is periodic with $0 \leq z < L$, the four dimensional gravitational constant G_4 is given by $G_4 = G_5/L$ and the total mass of the source is ML . Now, it is interesting to see if the asymptotic quantity j develops a new kinds of black string solution.

III. A STATIONARY BLACK STRING SOLUTION WITH MOMENTUM IN z -DIRECTION

Let us search for the metric whose source has the tension τ and the conserved momentum P . Then we know, from Eq. (6), that the leading corrections of the metric far away from the source are

$$h_{00} = \frac{4G_5 M(2-a)}{3\rho}, \quad h_{ij} = \delta_{ij} \frac{4G_5 M(1+a)}{3\rho}, \quad h_{44} = \frac{4G_5 M(1-2a)}{3\rho}, \quad h_{04} = \frac{2G_5 M(1-a)j}{\rho}. \quad (7)$$

The asymptotic form of the metric is

$$ds^2 \approx - \left(1 - \frac{4G_5 M(2-a)}{3\rho}\right) dt^2 + \left(1 + \frac{4G_5 M(1+a)}{3\rho}\right) (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) \\ + \left(1 + \frac{4G_5 M(1-2a)}{3\rho}\right) dz^2 + \frac{4G_5 M(1-a)j}{\rho} dt dz. \quad (8)$$

We have to find the solutions to the vacuum Einstein field equations which reduce to the asymptotic form of Eq. (8) at large ρ . We start with the ansatz

$$ds^2 = -F(\rho)[dt - X(\rho)dz]^2 + G(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + H(\rho)dz^2, \quad (9)$$

and substitute this form of metric into the vacuum Einstein field equation to derive differential equations for the four functions $F(\rho) = e^{2U}$, $G(\rho) = e^{2V}$, $H(\rho) = e^{2W}$, and $X(\rho)$:

$$U'' + (U' + V' + W' + \frac{2}{\rho})U' + \frac{(X')^2}{2}e^{2(U-W)} = 0, \quad (10) \\ U'' + 2V'' + W'' + (U')^2 + (W')^2 - (U' + W' - \frac{2}{\rho})V' - \frac{(X')^2}{2}e^{2(U-W)} = 0, \\ V'' + (U' + V' + W' + \frac{2}{\rho})V' + \frac{U' + V' + W'}{\rho} = 0, \\ W'' + (U' + V' + W' + \frac{2}{\rho})W' - \frac{(X')^2}{2}e^{2(U-W)} = 0, \\ X'' + (U' + V' + W' + \frac{2}{\rho})X' + 2(U' - W')X' = 0.$$

The above equations can be solved for $U + V + W$, $U + W$, and $e^{2(U-W)}X'$. After solving these, by summing the first two equations of Eq. (10), we obtain

$$D' + D^2 = \frac{3K^2(5 - b^2)}{(\rho^2 - K^2)^2}, \quad (11)$$

where $D = U' - W' + \frac{U' + V' + W'}{2} + \frac{1}{\rho}$ and b is a free parameter. A specific solution to Eq. (11) can be found to be $D_0 = (\sqrt{16 - 3b^2} + \rho)/(\rho^2 - K^2)$. Then, by setting $D = D_0 + 1/f$, we get a linear differential equation for f

$$f' - 2D_0 f = 1, \quad (12)$$

whose general solution can be obtained easily.

After changing the parameter $b = \frac{4}{\sqrt{3}\sqrt{1+\chi^2}}$, we have found the solution of the Einstein equation:

$$\begin{aligned} ds_{new}^2 = & -\frac{1}{1-q^2} \left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{2(\chi+1/\sqrt{3})}{\sqrt{1+\chi^2}}} \left(1 - q^2 \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}} \right) \left(dt + q \frac{1 - \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}}{1 - q^2 \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}} dz \right)^2 \\ & + \left| 1 - \frac{K^2}{\rho^2} \right|^2 \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + \frac{(1-q^2) \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{2(\chi-1/\sqrt{3})}{\sqrt{1+\chi^2}}}}{1 - q^2 \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}} dz^2, \end{aligned} \quad (13)$$

where the parameters K , χ , and q are related with the asymptotic parameters M , a , and j by

$$\begin{aligned} K &= \frac{G_5}{2\sqrt{3}}(1+a)M\sqrt{1+\chi^2}, \\ \chi &\equiv \chi(a, q) = \sqrt{3} \frac{1-a}{1+a} \frac{1-q^2}{1+q^2}, \\ q^2 &- 2j^{-1}q + 1 = 0. \end{aligned} \quad (14)$$

The solution is symmetric with respect to the following transformation of parameters:

$$q, \chi \longrightarrow q^{-1}, -\chi.$$

With this, we can restrict q to the finite region $-1 \leq q \leq 1$. In summary, the ranges of free parameters are given by

$$-\sqrt{3} < \chi \leq \sqrt{3}, \quad -\infty < K < \infty, \quad -1 \leq q \leq 1. \quad (15)$$

Because of the last equality of Eq. (14) and the restriction on the range of q , the relation between j and q becomes bijective. Therefore, we use q instead of j as a physical parameter which determines the conserved momentum. We have a unique stationary black string solution described by (χ, K, q) for a given asymptotic condition, (M, τ, P) .

IV. LOCAL EQUIVALENCE AND GLOBAL DIFFERENCE TO THE STATIC BLACK STRING SOLUTION

The curvature square of the metric (13) is

$$\begin{aligned} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = & \frac{192K^2}{\rho^6 (1 - K^2/\rho^2)^8} \left| \frac{1-K/\rho}{1+K/\rho} \right|^{\frac{8}{\sqrt{3}\sqrt{1+\chi^2}}} \left[\frac{K^4}{\rho^4} + 1 \right. \\ & \left. - \frac{4\sqrt{3}}{\sqrt{1+\chi^2}} \left(1 - \frac{4}{9(1+\chi^2)} \right) \frac{K}{\rho} \left(\frac{K^2}{\rho^2} + 1 \right) + 4 \left(1 + \frac{4}{3(1+\chi^2)} - \frac{8}{9(1+\chi^2)^2} \right) \frac{K^2}{\rho^2} \right]. \end{aligned} \quad (16)$$

Only at $|\chi| = 1/\sqrt{3}$, the terms inside the square bracket is factorized to $(1 - K/\rho)^4$. Then, in addition, the exponent of the absolute value becomes 4. For $|\chi| = 1/\sqrt{3}$, the curvature square at $\rho = K$ is finite, $3/(64K^4)$. However, the curvature square diverges at $\rho = K$ for $|\chi| \neq 1/\sqrt{3}$. The point $\rho = 0$ is a regular point with zero curvature. In fact, if one perform a coordinate transform $K/\rho \rightarrow \bar{\rho}/K$, one gets the same form of metric as Eq. (13), which implies that the region with $\rho \sim 0$ is also asymptotically flat. Note that this curvature square is independent of q . In addition, one can show that many curvature invariants of lower powers of $R_{\alpha\beta\mu\nu}$, $R_{\alpha\beta\mu\nu;\rho}$, and $R_{\alpha\beta\mu\nu;\rho\sigma}$ are independent of q

by using computer program such as Mathematica. At first glance, this seems to be a signal that the present solution is just a gauge artifact of the static black string solution (1). As we will see, this is not the case.

If two metrics are equivalent to each other, it implies that both the local and the global properties of the metrics are the same. The local equivalence of two metrics can be tested by comparing the Riemann curvature tensors and its covariant derivatives at each point of the spacetime. The global equivalence can be checked by analyzing the global structures of the spacetime.

For two equivalent metrics, there exists a general coordinate transformation which relates the two. Because it is not easy to find the explicit form of the transformation, we develop another method in the present case. Two equivalent metrics should be the same up to a Lorentz transformation at each event. At the present stationary metric, to check the local equivalence to the static black string metric, we investigate the q -dependence of the local geometric quantities. Once every local properties such as the Riemann tensors and their derivatives at an event are the same, the two metrics are equivalent at the event. Therefore, we examine the components of the Riemann tensor at a spacetime event $\mathcal{P} = (t, \rho, \theta, \phi, z)$ in a locally orthonormal frame of reference, given by the 1-form basis:

$$\omega^0 = [F^{1/2}(dt - Xdz)]_{\mathcal{P}}, \quad \omega^1 = [G^{1/2}d\rho]_{\mathcal{P}}, \quad \omega^2 = [G^{1/2}\rho d\theta]_{\mathcal{P}}, \quad \omega^3 = [G^{1/2}\rho \sin\theta d\phi]_{\mathcal{P}}, \quad \omega^4 = [H^{1/2}dz]_{\mathcal{P}}. \quad (17)$$

Since the locally orthonormal frame of reference is unique up to Lorentz transformation at an event \mathcal{P} , the equivalence of the metric (13) and (1) at \mathcal{P} can be checked. In Appendix A, by considering the ρ -dependent Lorentz boost along the z -direction,

$$\begin{pmatrix} x'^0 \\ x'^4 \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^4 \end{pmatrix}, \quad (18)$$

where x^i is a locally defined coordinate system in the orthonormal frame, we show that the components of the boosted Riemann tensor at \mathcal{P} for the new metric (13) become independent of q by choosing an appropriate boost parameter θ . In this sense, the new stationary solution is locally equivalent to the static black string solution (1).

However, an interesting inspection happens at $\rho = \rho_e$ in Eq. (A5) where $\tanh\theta = 1$. Around this point, the coordinate transformation is given by

$$\tanh\theta = \left[q \left| \frac{1 + K/\rho}{1 - K/\rho} \right|^{\frac{2\chi}{\sqrt{1+\chi^2}}} \right]^{\text{sign}(\rho - \rho_e)}. \quad (19)$$

Therefore, the coordinate transformation is continuous at ρ_e . Its first derivative,

$$\frac{\partial}{\partial\rho} \tanh\theta = -\frac{4\chi}{\sqrt{1+\chi^2}} \frac{\text{sign}(\rho - \rho_e)}{\left| \frac{1+K/\rho_e}{1-K/\rho_e} \right| (\rho_e - K)^2}, \quad (20)$$

is, however, discontinuous at ρ_e . Let us search for the effect of this discontinuity to the Riemann tensor. The effect on the components of the Riemann tensor appears in a form of combination:

$$(\cosh^2\theta + \sinh^2\theta)R_q - 2\cosh\theta\sinh\theta r_q = -\text{sign}(\rho - \rho_e)\sqrt{R_q^2 - r_q^2}, \quad (21)$$

where R_q and r_q are given in Appendix A. The right hand side of Eq. (21) is discontinuous at ρ_e for all $(q, \chi > 0)$ except for the pair given by the relation

$$q^{\frac{2\chi}{\sqrt{1+\chi^2}}} = \frac{\sqrt{3(1+\chi^2)} - 2 + 2\sqrt{1-3\chi^2}}{\sqrt{3(1+\chi^2)} + 2 - 2\sqrt{1-3\chi^2}}, \quad (22)$$

which comes from $R_q^2 = r_q^2$. This implies that the coordinate transformation (18) is pathological and the Riemann curvatures are equivalent only locally.

Since ρ_e presents only for $\chi > 0$, the local properties of the solution with $\chi \leq 0$ are exactly equivalent to the static black string solution. Now let us check the equivalence of the covariant derivatives of the Riemann tensor between the two solutions. For this purpose, we construct a Riemann normal coordinates around the point \mathcal{P} . In this coordinates the connection vanishes at the event \mathcal{P} . Therefore, the covariant derivatives of the Riemann tensor is given by the simple derivative with respect to ρ . Since the Riemann tensor in the frame is independent of q , their derivatives become automatically independent of q . In summary, the geometry of the new metric is locally equivalent to that of the static black string solution. However, the geometry of new solution with $\chi > 0$ is globally different from that of the static black string solution.

V. PROPERTIES OF THE STATIONARY SOLUTION

Next, let us investigate the global property of the metric such as the position of horizon. Consider the metric,

$$ds^2 = g_{tt}(\rho)dt^2 + 2g_{tz}(\rho)dtdz + g_{\rho\rho}(\rho)d\rho^2 + g_{zz}(\rho)dz^2, \quad (23)$$

where we have ignored θ and ϕ coordinates due to the spherical symmetry. The Killing vectors for t - and z -directions are $\xi_{(t)} \equiv \left(\frac{\partial}{\partial t}\right)_{\rho,z}$ and $\xi_{(z)} \equiv \left(\frac{\partial}{\partial z}\right)_{t,\rho}$. The scalar products of these Killing vectors generate the metric:

$$\xi_{(t)} \cdot \xi_{(t)} = g_{tt}, \quad \xi_{(t)} \cdot \xi_{(z)} = g_{tz}, \quad \xi_{(z)} \cdot \xi_{(z)} = g_{zz}. \quad (24)$$

The Killing vector of a stationary observer moving with velocity $v \equiv \frac{dz}{dt}$ relative to the asymptotic rest frame is

$$\xi_{(obs)} = \frac{\xi_{(t)} + v\xi_{(z)}}{|\xi_{(t)} + v\xi_{(z)}|} = \frac{\xi_{(t)} + v\xi_{(z)}}{(-g_{tt} - 2vg_{tz} - v^2g_{zz})^{1/2}}. \quad (25)$$

The stationary observer at a given ρ cannot have arbitrary velocity. Only the following value of v are allowed, for which the 4-velocity $\xi_{(obs)}$ lies inside the future light cone,

$$(\xi_{(t)} + v\xi_{(z)})^2 = g_{tt} + 2vg_{tz} + v^2g_{zz} < 0.$$

Thus, the velocity of stationary observers are constrained by,

$$\begin{aligned} v_{min} < v < v_{max}, \quad \text{for } g_{zz} > 0, \\ v < v_{min} \quad \text{or} \quad v > v_{max}, \quad \text{for } g_{zz} < 0, \end{aligned} \quad (26)$$

where

$$v_{min} = \bar{v} - \sqrt{\bar{v}^2 - g_{tt}/g_{zz}}, \quad v_{max} = \bar{v} + \sqrt{\bar{v}^2 - g_{tt}/g_{zz}}; \quad \bar{v} \equiv -\frac{g_{tz}}{g_{zz}} = -\frac{FX}{H - FX^2}. \quad (27)$$

Far from the black string, $v_{min} = -1$ and $v_{max} = 1$ with $g_{zz} > 0$. As ρ decreases, v_{min} increases. v_{min} becomes zero when $\frac{g_{tt}}{g_{zz}} = 0$, which is the static limit:

$$\rho_e = \frac{1 + q\sqrt{1+\chi^2}/(2\chi)}{1 - q\sqrt{1+\chi^2}/(2\chi)}K. \quad (28)$$

The static limit exists only for $\chi > 0$ since we restrict ourselves to the region with $\rho \geq K$. For $\chi < 0$, g_{zz} becomes negative for $\rho < \rho_t$ with

$$\rho_t = \frac{1 + q\sqrt{1+\chi^2}/(2|\chi|)}{1 - q\sqrt{1+\chi^2}/(2|\chi|)}K. \quad (29)$$

Therefore, the z coordinates becomes timelike there. For the time being, we consider the case $\chi > 0$ and defer $\chi \leq 0$ case.

Inside the static limit ($\rho = \rho_e$), all stationary observers must move along the black string with positive velocity. As $|q|$ increases, the position of the static limit, ρ_e , increases so that the ergosphere encompasses the entire space at $q = 1$. The allowed range of the velocity narrows down until the limits v_{min} and v_{max} coalesce at $\rho_H = K$ which satisfies

$$\sqrt{\bar{v}^2 - \frac{g_{tt}}{g_{zz}}} = \frac{1 - q^2}{\left|1 - q^2 \left(\frac{1-K/\rho_H}{1+K/\rho_H}\right)^{\frac{4\chi}{\sqrt{1+\chi^2}}}\right|} \left(\frac{1 - K/\rho_H}{1 + K/\rho_H}\right)^{\frac{2\chi}{\sqrt{1+\chi^2}}} = 0. \quad (30)$$

In the comoving frame with velocity \bar{v} , the position of the horizon is present when the metric g'_{tt} in the comoving frame vanishes:

$$g'_{tt} = g_{tt} + 2\bar{v}g_{tz} + \bar{v}^2g_{zz} = -\frac{FH}{H - FX^2} = \frac{1 - q^2}{1 - q^2 \left(\frac{1-K/\rho}{1+K/\rho}\right)^{\frac{4\chi}{\sqrt{1+\chi^2}}}} \left(\frac{1 - K/\rho}{1 + K/\rho}\right)^{\frac{2(\chi+1/\sqrt{3})}{\sqrt{1+\chi^2}}} = 0. \quad (31)$$

The solution of Eq. (31) exists for positive χ at $\rho = \rho_H = K$. Therefore, the solution (13) describes a black string solution without naked singularity. The velocity of the horizon is given by

$$v_H = - \left. \frac{g_{tz}}{g_{zz}} \right|_{\rho=K} = q, \quad 0 \quad \text{for } \chi = 0. \quad (32)$$

Let us now analyze for the case $\chi < 0$. Since the z -coordinate becomes time-like for $\rho < \rho_t$, the analysis for a stationary observer is not appropriate to probe an event horizon. Consider the coordinate transformation from $(t, \rho, z) \rightarrow (\tau, \rho, z)$ with $\tau = vt + z$, where v is a constant velocity. The metric (23) now becomes

$$ds^2 = \frac{g_{tt}}{v^2} d\tau^2 + \frac{2}{v} \left(g_{tz} - \frac{g_{tt}}{v} \right) d\tau dz + \left(\frac{g_{tt}}{v^2} - \frac{2g_{tz}}{v} + g_{zz} \right) dz^2 + g_{\rho\rho} d\rho^2. \quad (33)$$

At a point $\mathcal{Q} = (\tau, \rho, z)$, we may find a locally orthogonal metric by setting the velocity $v = \frac{g_{tt}(\rho)}{g_{tz}(\rho)}$. Now, the metric at the point \mathcal{Q} becomes

$$ds^2|_{\mathcal{Q}} = \frac{g_{tz}^2(\rho)}{g_{tt}(\rho)} d\tau^2 + \left(g_{zz}(\rho) - \frac{g_{tz}^2(\rho)}{g_{tt}(\rho)} \right) dz^2 + g_{\rho\rho}(\rho) d\rho^2. \quad (34)$$

Note that the zz components of the metric at \mathcal{Q} ,

$$g_{zz}^{\mathcal{Q}} \equiv g_{zz}(\rho) - \frac{g_{tz}^2(\rho)}{g_{tt}(\rho)} = H = \frac{(1 - q^2) \left| \frac{1-K/\rho}{1+K/\rho} \right|^{\frac{2(|\chi|+1/\sqrt{3})}{\sqrt{1+\chi^2}}}}{1 - q^2 \left| \frac{1-K/\rho}{1+K/\rho} \right|^{\frac{4|\chi|}{\sqrt{1+\chi^2}}}},$$

is positive definite for all \mathcal{Q} if $\chi \leq 0$. This value vanishes at point $\mathcal{Q}_K = (\tau, K, z)$. We may write the geodesic equation in this frame of reference. For fixed z , the radially outgoing light-like geodesic satisfies

$$\left. \frac{d\rho}{d\tau} \right|_{\mathcal{Q}} = \sqrt{-\frac{g_{tz}^2(\rho)}{g_{tt}(\rho)g_{\rho\rho}(\rho)}} = \sqrt{\frac{q^2}{1 - q^2} \frac{\left(1 - \left| \frac{1-K/\rho}{1+K/\rho} \right|^{\frac{4|\chi|}{\sqrt{1+\chi^2}}} \right)^2}{1 - q^2 \left| \frac{1-K/\rho}{1+K/\rho} \right|^{\frac{4|\chi|}{\sqrt{1+\chi^2}}}}} \left(1 + \frac{K}{\rho} \right)^{-1 - \frac{\sqrt{3}+\chi}{\sqrt{1+\chi^2}}} \left(1 - \frac{K}{\rho} \right)^{-1 + \frac{\sqrt{3}+\chi}{\sqrt{1+\chi^2}}}. \quad (35)$$

The term inside the square root does not vanish for $\rho \geq K$ and $q \neq 0$. At point \mathcal{Q}_K , the velocity (35) vanishes only for $\chi > -1/\sqrt{3}$. Therefore, the event horizon forms for $\chi > -1/\sqrt{3}$ at $\rho = K$. This result is exactly the same as that of the static black string solution (1). With this analysis, we may argue that the structure of the event horizon for both metrics are the same. Therefore, the solution (13) with $(K, \chi \leq 0, q)$ are equivalent to that with $(K, \chi, 0)$ up to coordinate transformation.

VI. SUMMARY AND DISCUSSIONS

We have obtained a new general stationary black string solution with arbitrary tension and momentum along the fifth coordinate which has non-trivial g_{04} component decreasing as $1/\rho$ for large radial distance ρ . The new solution shows an exact local equivalence to the static black string solution, but the solution with $\chi > 0$ is globally different from the static one except for the parameters satisfying Eq. (22).

We briefly have analyzed some physical properties of the metric such as an event horizon and a static limit. Let us briefly discuss the area of the horizon. The 2-dimensional surface area for a fixed ρ , z , and t can be obtained by integrating the θ and ϕ coordinates,

$$S(\rho) = 4\pi\rho^2 G(\rho) = 4\pi\rho^2 \left| 1 - \frac{K}{\rho} \right|^{2 - \frac{4}{\sqrt{3(1+\chi^2)}}} \left| 1 + \frac{K}{\rho} \right|^{2 + \frac{4}{\sqrt{3(1+\chi^2)}}}. \quad (36)$$

The area for a given ρ diverges for $|\chi| < \frac{1}{\sqrt{3}}$ and vanishes for $|\chi| > \frac{1}{\sqrt{3}}$ as $\rho \rightarrow K$. Only at $|\chi| = \frac{1}{\sqrt{3}}$, a finite horizon area is obtained:

$$S = 64\pi K^2. \quad (37)$$

We consider only for $\chi \geq 0$. The distance $|\delta z(\rho)|$ corresponding to a unit length in z -direction for an asymptotic observer in a constant t surface is measured by

$$|\delta z|_t^2 = g_{zz} = \frac{1 - q^2 \left| \frac{1 - K/\rho}{1 + K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}}{1 - q^2} \left| \frac{1 - K/\rho}{1 + K/\rho} \right|^{\frac{2(1/\sqrt{3} - \chi)}{\sqrt{1+\chi^2}}}. \quad (38)$$

This behaves as

$$\lim_{\rho \rightarrow K} |\delta z|_t^2 = \frac{1}{1 - q^2} \left| \frac{1 - K/\rho}{1 + K/\rho} \right|^{\frac{2(1/\sqrt{3} - \chi)}{\sqrt{1+\chi^2}}}. \quad (39)$$

Therefore, δz at the horizon vanishes for $\chi < 1/\sqrt{3}$, and diverges for $\chi > 1/\sqrt{3}$. Only at $\chi = 1/\sqrt{3}$ a finite measure along z -coordinate at the horizon are defined.

The surface area of the black string for unit length of z with respect to an asymptotic observer is given by multiplying $S(\rho)$ and δz :

$$A = S(\rho)|\delta z| = 4\pi\rho^2 \left(\frac{1 - q^2 \left| \frac{1 - K/\rho}{1 + K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}}{1 - q^2} \right)^{1/2} \left| 1 - \frac{K}{\rho} \right|^{2 - \frac{\sqrt{3} + \chi}{\sqrt{1+\chi^2}}} \left| 1 + \frac{K}{\rho} \right|^{2 + \frac{\sqrt{3} + \chi}{\sqrt{1+\chi^2}}}. \quad (40)$$

The exponent $2 - (\sqrt{3} + \chi)/\sqrt{1 + \chi^2}$ is a non-negative function which vanishes only at $\chi = 1/\sqrt{3}$. Therefore, The area at the horizon $\rho = K$ vanishes for $0 \leq \chi \neq 1/\sqrt{3}$. For $\chi = 1/\sqrt{3}$, the area becomes $64\pi K^2/(1 - q^2)$. Since the area of the horizon is proportional to the entropy of the black string, the black string with $\chi \neq \frac{1}{\sqrt{3}}$ has zero entropy.

In the stability context, this implies the black string solution will be unstable.

In this paper, we write the surface $\rho = K$ as an event horizon. However, the exact identity of the surface $\rho = K$ is ambiguous yet and is under investigation by Kang et. al. [13]. As we have summarized in Sec. II, the properties of the metric at $\rho = K$ becomes singular for $|\chi| \neq 1/\sqrt{3}$. Because the singularity is located at the same position as the horizon, its physical properties are different from the usual case.

The present metric provides an interesting test ground of the Mach's principle (Inertia generates gravity) for linear motions. In a literal sense, the linear motion should generate a velocity frame dragging effect if the Mach's principle is correct. According to the present result of the metric, the linear motion along the fifth coordinate does not develop a local difference of spacetime. However, it changes the global structures of spacetime for some parameter range. Since the presence of frame dragging by inertial motion is a controversial subject today, this result provides an interesting inspection.

Let us discuss how the global difference of the present solution from the static black string solution can be understood qualitatively. As we know in the Aharonov-Bohm effect, the electro-magnetic gauge potentials with different fluxes outside the flux tube are locally equivalent but physically different, which comes from global effects. In the present case, the situation is quite similar. The metrics with different momenta are locally equivalent but globally different. The conserved momentum plays a role of the magnetic flux in Aharonov-Bohm effect. The deep implications of this result are under investigation.

Acknowledgments

We are very grateful to Gungwon Kang for helpful discussions. This work was supported by Daejin University Research Grants in 2006.

APPENDIX A: LORENTZ BOOST OF THE RIEMANN TENSOR AN AN EVENT \mathcal{P}

The non-vanishing components of Riemann tensor in the 1-form basis (17) at the event \mathcal{P} become

$$\begin{aligned}
R_{0101} &= R_1 + R_q, & R_{0114} &= R_{1401} = r_q, & R_{1414} &= -R_1 + R_q, \\
R_{0202} &= R_{0303} = R_2 - \frac{1}{2}R_q, & R_{0224} &= R_{2402} = R_{0334} = R_{3403} = -\frac{r_q}{2}, \\
R_{2424} &= R_{3434} = -R_2 - \frac{1}{2}R_q, & R_{0404} &= \frac{1-3\chi^2}{1+\chi^2} \frac{4\kappa^2}{3\rho^4(1-K^2/\rho^2)^4} \left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}}, \\
R_{1212} &= R_{1313} = -\frac{4}{\sqrt{3(1+\chi^2)}} \frac{\kappa(1-\sqrt{3(1+\chi^2)}K/\rho + K^2/\rho^2)}{\rho^3(1-K^2/\rho^2)^4} \left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}}, \\
R_{2323} &= \frac{4}{3(1+\chi^2)} \frac{K(2K/\rho - \sqrt{3(1+\chi^2)})(\sqrt{3(1+\chi^2)}K/\rho - 2)}{\rho^3(1-K^2/\rho^2)^4} \left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}},
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= \frac{4K\chi}{\sqrt{1+\chi^2}} \frac{\left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}}}{\rho^3(1-K^2/\rho^2)^4} \left[\frac{4K}{\rho} - \frac{4}{\sqrt{3(1+\chi^2)}} \left(\frac{K^2}{\rho^2} + 1 \right) \right], \\
R_2 &= \frac{2K}{\sqrt{3(1+\chi^2)}} \frac{\left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}}}{\rho^3(1-K^2/\rho^2)^4} \left(1 - \frac{4}{\sqrt{3(1+\chi^2)}} \frac{K}{\rho} + \frac{K^2}{\rho^2} \right), \\
R_q &= -\frac{4K\chi}{\sqrt{1+\chi^2}} \frac{\left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}}}{\rho^3(1-K^2/\rho^2)^4} \left(1 - \frac{4}{\sqrt{3(1+\chi^2)}} \frac{K}{\rho} + \frac{K^2}{\rho^2} \right) \frac{1+q^2 \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}}{1-q^2 \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}}, \\
r_q &= \frac{8K\chi}{\sqrt{1+\chi^2}} \frac{\left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}}}{\rho^3(1-K^2/\rho^2)^4} \left(1 - \frac{4}{\sqrt{3(1+\chi^2)}} \frac{K}{\rho} + \frac{K^2}{\rho^2} \right) \frac{q \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{2\chi}{\sqrt{1+\chi^2}}}}{1-q^2 \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4\chi}{\sqrt{1+\chi^2}}}}.
\end{aligned} \tag{A1}$$

Now consider a Lorentz boost along the z -direction at the event \mathcal{P} :

$$\begin{pmatrix} x'^0 \\ x'^4 \end{pmatrix} = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^4 \end{pmatrix} \tag{A2}$$

where x^i is a coordinate system in the orthonormal frame at \mathcal{P} . In the boosted coordinates x'^i , the components of Riemann tensor transform to be

$$\begin{aligned}
R'_{0101} &= R_1 + (\cosh^2 \theta + \sinh^2 \theta) R_q - 2 \cosh \theta \sinh \theta r_q, \\
R'_{1414} &= -R_1 + (\cosh^2 \theta + \sinh^2 \theta) R_q - 2 \cosh \theta \sinh \theta r_q, \\
R'_{0114} &= (\cosh^2 \theta + \sinh^2 \theta) r_q - 2 \cosh \theta \sinh \theta R_q, \\
R'_{0202} &= R_2 - \frac{1}{2} [(\cosh^2 \theta + \sinh^2 \theta) R_q - 2 \cosh \theta \sinh \theta r_q], \\
R'_{0224} &= -\frac{1}{2} [(\cosh^2 \theta + \sinh^2 \theta) r_q - 2 \cosh \theta \sinh \theta R_q], \\
R'_{2424} &= -R_2 - \frac{1}{2} [(\cosh^2 \theta + \sinh^2 \theta) r_q - 2 \cosh \theta \sinh \theta R_q], \\
R'_{0404} &= R_{0404}.
\end{aligned} \tag{A3}$$

By choosing the boost parameter to satisfy

$$\tanh \theta + \frac{1}{\tanh \theta} = \frac{2R_q}{r_q} = \frac{1}{q} \left| \frac{1+K/\rho}{1-K/\rho} \right|^{-\frac{2\chi}{\sqrt{1+\chi^2}}} + q \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{2\chi}{\sqrt{1+\chi^2}}},$$

we have $R'_{0114} = 0 = R'_{0224}$. The unique solution of the equation is

$$\tanh \theta = \left[q \left| \frac{1 + K/\rho}{1 - K/\rho} \right|^{\frac{2\chi}{\sqrt{1+\chi^2}}} \right]^{\text{sign}(\rho - \rho_e)}. \quad (\text{A4})$$

where ρ_e is the position of the ergosphere of the metric (13),

$$\rho_e = \frac{1 + |q|^{\frac{\sqrt{1+\chi^2}}{2\chi}}}{1 - |q|^{\frac{\sqrt{1+\chi^2}}{2\chi}}} K. \quad (\text{A5})$$

The q -dependent part of components of Riemann tensor in Eq. (A3) is of the same form:

$$\begin{aligned} (\cosh^2 \theta + \sinh^2 \theta) R_q - 2 \cosh \theta \sinh \theta r_q &= -\text{sign}(\rho - \rho_e) \sqrt{R_q^2 - r_q^2} \\ &= \text{sign}(\rho - \rho_e) \frac{4K\chi}{\sqrt{1+\chi^2}} \left| \frac{1+K/\rho}{1-K/\rho} \right|^{\frac{4}{\sqrt{3}\sqrt{1+\chi^2}}} \left| 1 - \frac{4}{\sqrt{3(1+\chi^2)}} \frac{K}{\rho} + \frac{K^2}{\rho^2} \right|, \end{aligned} \quad (\text{A6})$$

where the last expression of Eq. (A6) is independent of q . Therefore, the components of Riemann tensor (A3) at the event \mathcal{P} in the transformed coordinate system becomes independent of q .

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